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INCOMPRESSIBLE FLOW PAST A SINUSOIDAL WALL OF FINITE AMPLITUDE

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SUMMARY

The problem of the present paper has been chosen for the purpose of exhibiting some of the possible mathematical troubles that may occur in the iterative procedures so much in use in present-day aerodynamics. First, it is shown that the example of incompressible flow past a sinusoidal wall of finite amplitude should properly be treated in the plane of the velocity potential and the stream function rather than in the physical flow plane. Then, two contrasting iterative procedures are utilized for the solution of this particular problem. One is the well-known small-disturbance method in which the physical-plane coordinates are determined in the form of Fourier series whose coefficients are analytically developed as series in ascending powers of the amplitude. In general, this method precludes any discussion of convergence, the tacit assumption being that no mathematical limitation intervenes before the solution ceases to be valid because of some physical reason. The other procedure is to state the problem in the form of an integral equation whose solution can be found by a process of successive approximations. The convergence of this method can usually be judged when the difference between any two successive approximations is deemed negligible. An included numerical example serves to emphasize the superiority of the integral-equation approach over the small-disturbance method.

INTRODUCTION

In the nonlinear treatment of stationary compressible flows, the methods utilized very often involve developments in series and successive approximations. One of the most widely used of these methods is the Prandtl-Busemann small-disturbance procedure for the calculation of two-dimensional compressible flow past a thin profile. According to this method, the velocity potential or the stream function is developed in powers of the thickness coefficient with the thin profile placed at a vanishingly small angle of attack in a uniform stream of Mach number less than unity. Thus far, no strictly mathematical investigation has been made on the convergence of this small-disturbance procedure. Indeed, the labor increases so rapidly with the order of the approximation that usually only two or three steps can be calculated; thus, any possible rigorous statement on the question of convergence is eliminated. The first step in this method of iteration being the undisturbed stream itself, the decisive element in the convergence of the process is the tacit assumption that no purely mathematical limitation sets in before the breakdown of the flow due to some physical reason (for example, the attainment of sonic velocity at some point in the field of flow). That such mathematical difficulties can appear, even without the complication of compressibility, is illustrated in the problem treated in the present paper, namely, incompressible flow past a sinusoidal wall of finite amplitude. This solid boundary has been chosen chiefly because the basic assumptions of the small-disturbance method are adhered to, there being no stagnation points or points of infinite velocity in the field of flow, and because numerous iteration steps can be achieved without undue labor. Moreover, it was found possible to solve the problem independently by means of an integral-equation approach much in the manner of the arbitrary-airfoil theory of Theodorsen and Garrick (ref. 1). A comparison of these two modes of solution clearly reveals basic weaknesses in the small-disturbance method when it is utilized for the purpose of approximating analytically the exact solution of a flow problem.

It may be mentioned that the sinusoidal wall in two dimensions and its axisymmetric counterpart, the corrugated cylinder, have in the past been of considerable aid in the solution of various problems in aerodynamics (refs. 2 to 7). It is believed that the material of the present paper may be found useful in the treatment of such diverse problems as panel flutter and the hydrodynamic theory of water waves of finite amplitude.

ANALYSIS

General Formulas

When the Prandtl-Busemann small-disturbance method is utilized to obtain the two-dimensional incompressible flow past the wavy wall

$$y = \epsilon \cos x$$

the complex potential of the flow w can be represented in the following fashion:

$$w = \phi + i\psi = z - \frac{1}{2} iB_0(\epsilon) - i \sum_{n=1}^{\infty} B_n(\epsilon)e^{inz} \tag{1}$$

where

x,y nondimensional rectangular Cartesian coordinates, radians

$z = x + iy$

ϵ amplitude of wavy wall, radians

ϕ nondimensional velocity potential

ψ nondimensional stream function

$B_n(\epsilon)$ real quantities presumed to be expansible in ascending powers of ϵ

and where U, the undisturbed stream velocity, and λ, the wave length in radians, are utilized as units of velocity and length, respectively.

According to equation (1), it is tacitly assumed that w is an analytic function of z along and above the solid wavy wall. However, the covergence of the series appearing on the right-hand side is confined to the broadest strip which is parallel to the x-axis and is devoid of singular points of w (ref. 8). Thus, in order for this series to represent w on and above the sinusoidal boundary, w must be free of singularities in a strip containing the boundary. There is no assurance, however, that such singularities do not appear in the portion of the strip below the wavy wall as soon as the solid boundary departs from a straight line. Therefore, the solution in the form of equation (1) is not always valid and may even be divergent for $\epsilon > 0$.

The rather surprising statement that the small-disturbance iteration method cannot always be applied in the physical plane to the problem of flow past a sinusoidal boundary poses the question of what is the correct mathematical approach to this well-defined flow problem. The answer is simply that the problem should be treated in the $\phi\psi$-plane, where the sinusoidal boundary $y = \epsilon \cos x$ is replaced by the straight line $\psi = 0$. Equation (1) is then replaced by

$$z = w + \frac{1}{2} iA_0(\epsilon) + i \sum_{n=1}^{\infty} A_n(\epsilon)e^{inw} \tag{2}$$

where the right-hand side represents a complex function of w which is now analytic in the entire upper half-plane. The coefficients $A_n(\epsilon)$ are real quantities presumed to be expansible in ascending powers of ϵ. No boundary condition need be stated in the $\pm\phi$ directions because the solution is periodic in ϕ; whereas for $\psi = \infty$, the complex velocity is, as it must be, $\frac{dw}{dz} = 1$. Expressed in real form, equation (2) is equivalent to the pair of equations

$$x = \phi - \sum_{n=1}^{\infty} A_n(\epsilon)e^{-n\psi}\sin n\phi \tag{3a}$$

and

$$y = \psi + \frac{1}{2} A_0(\epsilon) + \sum_{n=1}^{\infty} A_n(\epsilon)e^{-n\psi}\cos n\phi \tag{3b}$$

At the solid boundary, $\psi = 0$; therefore,

$$x = \phi - \sum_{n=1}^{\infty} A_n(\epsilon)\sin n\phi \tag{4a}$$

and

$$\epsilon \cos x = \frac{1}{2} A_0(\epsilon) + \sum_{n=1}^{\infty} A_n(\epsilon)\cos n\phi \tag{4b}$$

From equation (4b) (since $\epsilon \cos x$ is an even periodic function of ϕ),

$$A_n(\epsilon) = \frac{2}{\pi} \epsilon \int_0^{\pi} \cos x \cos n\phi \, d\phi \qquad (n = 0, 1, 2, \ldots \infty)$$

Inserting the expression for x given by equation (4a) into this equation yields the following formula:

$$A_n(\epsilon) = \frac{2}{\pi} \epsilon \int_0^{\pi} \cos n\phi \cos\left[\phi - \sum_{m=1}^{\infty} A_m(\epsilon)\sin m\phi\right] d\phi \tag{5}$$

If the quantities $A_n(\epsilon)$ are assumed to be expansible in a series of ascending powers of ϵ, equation (5) represents the recursion formula for the determination of the coefficients of these series. Preliminary calculations show that the $A_n(\epsilon)$ can be expressed as

$$\frac{1}{2} A_0(\epsilon) = \sum_{m=1}^{\infty} a_{0m}\epsilon^{2m} \tag{6a}$$

and

$$A_n(\epsilon) = \sum_{m=0}^{\infty} a_{nm}\epsilon^{n+2m} \qquad (n = 1, 2, \ldots \infty) \tag{6b}$$

where the a_{nm} are numerical coefficients uniquely determined by repeated use of equation (5). Thus, the initial step in the determination of the a_{nm} is to obtain a_{10}, the coefficient of the first term of the series for $A_1(\epsilon)$. From equation (5), insofar as the first power of ϵ is concerned,

$$A_1(\epsilon) = \frac{2}{\pi} \epsilon \int_0^{\pi} \cos^2\phi \, d\phi = \epsilon$$

or

$$a_{10} = 1$$

The second step is to obtain the first term in the power-series development of $A_2(\epsilon)$. Thus,

$$A_2(\epsilon) = \frac{2}{\pi} \epsilon \int_0^{\pi} \cos 2\phi \cos(\phi - \epsilon \sin \phi) d\phi$$

This expression for $A_2(\epsilon)$ is easily evaluated with the aid of the well-known Jacobi expansions in series of Bessel functions; namely,

$$\cos(\epsilon \sin \phi) = J_0(\epsilon) + 2 \sum_{n=1}^{\infty} J_{2n}(\epsilon)\cos 2n\phi$$

$$\sin(\epsilon \sin \phi) = 2 \sum_{n=0}^{\infty} J_{2n+1}(\epsilon)\sin(2n + 1)\phi$$

and the power-series expression for Bessel functions of the first kind

$$J_n(\epsilon) = \sum_{k=0}^{\infty} (-1)^k \frac{\epsilon^{2k+n}}{2^{2k+n}k!(k + n)!}$$

Insofar as the second power of ϵ is concerned, the result is

$$A_2(\epsilon) = -\frac{1}{2}\epsilon^2$$

or

$$a_{20} = -\frac{1}{2}$$

Similarly,

$$A_0(\epsilon) = \frac{2}{\pi}\epsilon \int_0^{\pi} \cos(\phi - \epsilon \sin \phi)d\phi$$

or

$$\frac{1}{2} A_0(\epsilon) = \frac{1}{2}\epsilon^2$$

and

$$a_{01} = \frac{1}{2}$$

In this manner, the power-series developments for $A_n(\epsilon)$ can be systematically constructed. Table I lists values of the coefficients a_{nm} sufficient in number to determine the solution, equations (3), to the order ϵ^9.

Examination of table I shows that the general formula for the set of coefficients a_{n0} can be written as

$$a_{n0} = -(-1)^n \frac{n^{n-1}}{2^{n-1}\, n!} \qquad (n = 1, 2, \ldots \infty) \tag{7}$$

and that

$$\sum_{n=1}^{\infty} A_{2n-1}(\epsilon) = \epsilon \tag{8}$$

Moreover, from equations (4), it can be seen that

$$\frac{1}{2} A_0(\epsilon) + \sum_{n=1}^{\infty} A_n(\epsilon) = \epsilon$$

Thus, with the aid of equation (8), it follows that

$$\frac{1}{2} A_0(\epsilon) = -\sum_{n=1}^{\infty} A_{2n}(\epsilon) \tag{9}$$

From considerations of symmetry, the area under the boundary curve for one wave length is zero; that is,

$$\int_0^{2\pi} y \, dx = 0$$

Then, by means of equations (4),

$$\int_0^{2\pi}\left[\frac{1}{2}A_0(\epsilon)+\sum_{n=1}^{\infty}A_n(\epsilon)\cos n\phi\right]\left[1-\sum_{m=1}^{\infty}mA_m(\epsilon)\cos m\phi\right]d\phi=0$$

or

$$\frac{1}{2}A_0(\epsilon)=\frac{1}{2}\sum_{n=1}^{\infty}nA_n^2(\epsilon)$$

Comparison with equation (9) shows that

$$\frac{1}{2}\sum_{n=1}^{\infty}nA_n^2(\epsilon)=-\sum_{n=1}^{\infty}A_{2n}(\epsilon) \tag{10}$$

Linearized Case

In the linearized case in which terms involving only the first power of ϵ are retained, equations (3) become

$$x=\phi-\epsilon e^{-\psi}\sin\phi \tag{11a}$$

and

$$y=\psi+\epsilon e^{-\psi}\cos\phi \tag{11b}$$

It is interesting to note that equation (11a) is similar to one that Bessel discussed in connection with Kepler's eccentric-anomaly problem (ref. 9). Corresponding to Bessel's solution,

$$\phi=x+2\sum_{n=1}^{\infty}\frac{1}{n}J_n(n\epsilon e^{-\psi})\sin nx \tag{12}$$

The process by which this result is obtained gives no information as to the conditions under which the expansion on the right-hand side is possible. Therefore, until the interval of convergence has been determined, it is not legitimate to discard terms involving powers of ϵ higher than the first. However, the series in equation (12) is a Kapteyn series which converges rapidly when $\epsilon e^{-\psi} < 1$ and is convergent even for $\epsilon e^{-\psi} = 1$. Thus, for the linearized case where $\epsilon \ll 1$, only terms involving the first power of ϵ need be retained. The reversion of equations (11) then yields

$$\left.\begin{aligned} \phi &= x + \epsilon e^{-y}\sin x \\ \psi &= y - \epsilon e^{-y}\cos x \end{aligned}\right\} \tag{13}$$

which are in agreement with the results obtained when the linearized case is treated in the plane of flow. This result is not in contradiction with the statements made with regard to equation (1), because for the case of vanishingly small ϵ, the boundary condition of zero normal velocity is satisfied along the real axis $y = 0$.

Numerical Example of a Wavy Wall of Finite Amplitude

In choosing a numerical example, it is kept in mind that ϵ should be less than one but yet should correspond to a wavy wall of moderate amplitude. Thus, the value $\epsilon = 0.70$ corresponds to a wavy wall whose amplitude is about one-ninth of the wave length. The conversion of results obtained in the $\phi\psi$-plane to results in the physical flow plane is easily made by means of equations (3). Indeed, these equations are particularly well-suited for the construction of streamlines, which is usually a long and tedious computation. Equation (3a) shows that ϕ is a continuously increasing function of x, such that the effect of increasing x by 2π is to increase ϕ by 2π. Moreover, $\sum_{n=1}^{\infty} A_n(\epsilon)e^{-n\psi}\sin n\phi$ is an odd periodic function of x and, hence, calculations need be made only for the interval $0 \leqq x \leqq \pi$.

The following formulas are to be utilized for the calculation of fluid speed and pressure coefficient at the surface of the wavy wall. From equation (2),

$$\frac{dw}{dz} = u - iv = \frac{1}{1 - \sum_{n=1}^{\infty} nA_n(\epsilon)e^{-n\psi}\cos n\phi - i\sum_{n=1}^{\infty} nA_n(\epsilon)e^{-n\psi}\sin n\phi}$$

where $u = \frac{\partial\phi}{\partial x}$ and $v = \frac{\partial\phi}{\partial y}$ are the components of the fluid velocity in the direction of the x- and y-axes, respectively.

The magnitude of the fluid velocity at the solid boundary $\psi = 0$ is given by

$$q^2 = \frac{1}{\left[1 - \sum_{n=1}^{\infty} nA_n(\epsilon)\cos n\phi\right]^2 + \left[\sum_{n=1}^{\infty} nA_n(\epsilon)\sin n\phi\right]^2} \tag{14}$$

In particular, the maximum and minimum speeds occur at $\phi = 0$ and $\phi = \pi$, respectively; thus,

$$q_{max} = \frac{1}{1 - \sum_{n=1}^{\infty} nA_n(\epsilon)} \tag{15a}$$

and

$$q_{min} = \frac{1}{1 - \sum_{n=1}^{\infty} (-1)^n nA_n(\epsilon)} \tag{15b}$$

The pressure coefficient C_p, obtained by means of Bernoulli's theorem, is given by

$$C_p = 1 - q^2 \tag{16}$$

The coefficients $A_n(\epsilon)$ have been developed as power series in ϵ in the form given by equations (6). Actually, there is no basic reason why these coefficients should be expressed in this fashion. In fact, this power-series representation of the coefficients $A_n(\epsilon)$, although suitable enough for wavy walls of small amplitude ($\epsilon \ll 1$), may not be suitable for walls of even moderate amplitude ($\epsilon < 1$). Moreover, it would appear that for walls of small amplitudes the linearized results given by equations (11) or (13) suffice and that the labor expended in obtaining the coefficients a_{nm} is hardly worthwhile. Therefore, in order to obtain results appropriate for values of the amplitude ϵ of the order of unity, for example, some system of approximation other than power-series development in ϵ must be adopted. The most obvious one is that first introduced by Stokes in connection with his work on the theory of oscillatory waves (p. 317 of ref. 10). Stokes' method of approximation is to proceed according to powers of the first coefficient $A_1(\epsilon)$. That this approach is an independent one can be seen from the recursion formula (eq. (5)) since ϵ appearing therein can be replaced by the sum $\sum_{n=1}^{\infty} A_{2n-1}(\epsilon)$ (see eq. (8)). However, the coefficients a_{nm} already listed in table I can be utilized to obtain the expressions for $A_2(\epsilon)$, $A_3(\epsilon)$, . . . as series in ascending powers of $A_1(\epsilon)$. They are as follows:

$$\left.\begin{aligned}
A_2 &= -\frac{1}{2} A_1{}^2 + \frac{1}{12} A_1{}^4 + \frac{1}{64} A_1{}^6 + \frac{1}{15 \times 2^7} A_1{}^8 + \dots \\
A_3 &= \frac{3}{8} A_1{}^3 - \frac{53}{384} A_1{}^5 - \frac{7}{512} A_1{}^7 + \frac{1529}{45 \times 2^{13}} A_1{}^9 + \dots \\
A_4 &= -\frac{1}{3} A_1{}^4 + \frac{373}{15 \times 2^7} A_1{}^6 + \frac{1}{9 \times 2^9} A_1{}^8 + \dots \\
A_5 &= \frac{125}{384} A_1{}^5 - \frac{6031}{45 \times 2^9} A_1{}^7 + \frac{329}{3 \times 2^{12}} A_1{}^9 + \dots \\
A_6 &= -\frac{27}{80} A_1{}^6 + \frac{112039}{315 \times 2^{10}} A_1{}^8 + \dots \\
A_7 &= \frac{16807}{45 \times 2^{10}} A_1{}^7 - \frac{525603}{35 \times 2^{15}} A_1{}^9 + \dots \\
A_8 &= -\frac{128}{315} A_1{}^8 + \dots \\
A_9 &= \frac{531441}{35 \times 2^{15}} A_1{}^9 - \dots
\end{aligned}\right\} \tag{17}$$

From equation (8), it follows that

$$A_1 + \frac{3}{8} A_1^3 + \frac{3}{16} A_1^5 + \frac{823}{9216} A_1^7 + \frac{26551}{45 \times 2^{14}} A_1^9 + \ldots = \epsilon \qquad (18)$$

and from equation (9), it follows that

$$\frac{1}{2} A_0 = \frac{1}{2} A_1^2 + \frac{1}{4} A_1^4 + \frac{49}{384} A_1^6 + \frac{179}{3072} A_1^8 + \ldots \qquad (19)$$

Note that with the exception of the first coefficient, where the values are the same, the coefficients of the power series in A_1 shown in equations (17) are consistently and substantially less than the corresponding coefficients of the power series in ϵ. Moreover, for a given value of ϵ, the corresponding value of A_1 is less. Thus, for $\epsilon = 0.70$ the value of A_1, obtained from equation (18), is 0.60. The values of the other coefficients calculated by means of equations (17) and (19) are as follows:

$$A_2 = -0.169 \qquad A_3 = 0.070 \qquad A_4 = -0.037$$

$$A_5 = 0.021 \qquad A_6 = -0.010 \qquad A_7 = 0.006$$

$$\frac{1}{2} A_0 = 0.220$$

These values of A_n when introduced into equations (4) reproduce well the boundary curve $y = 0.70 \cos x$; other streamlines (with $\psi > 0$) will be even more accurately computed because of the presence of the factor $e^{-n\psi}$ in equations (3). Figure 1 shows the wavy wall for $\epsilon = 0.70$ and the streamlines which were calculated by means of equations (3) for values corresponding to $\psi = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}$, and 2. However, when these values of A_n are inserted into equations (14) and (16) in order to obtain the fluid speed and the pressure coefficient at the surface, the resulting points are scattered in a manner which indicates a slow convergence of the derived series which appear in these equations.

It is clear that, for wavy walls of even moderate amplitude, further analytical development of the coefficients $A_n(\epsilon)$ is of little avail because on the one hand a large number of terms of the series would have to be taken, and because on the other hand the labor of the approximation increases inordinately with the order of the terms. Therefore, some independent method must be found which avoids the necessity of determining analytically an infinite number of coefficients. This statement may very well apply to flow problems in general when the exact solution is to be approximated in some manner by an iterative procedure. For the present problem, such a method is described in the following section.

Integral-Equation Approach

From equation (4b),

$$A_n(\epsilon) = \frac{2}{\pi} \epsilon \int_0^{\pi} \cos x \cos n\phi \, d\phi \qquad (n = 0; 1, 2, \ldots \infty)$$

The evaluation of the infinite number of coefficients $A_n(\epsilon)$ can be made to depend upon a single equation which is obtained by eliminating $A_n(\epsilon)$ from equation (4a). Thus,

$$x = \phi - \frac{2}{\pi} \epsilon \sum_{n=1}^{\infty} \sin n\phi \int_0^{\pi} \cos x' \cos n\phi' \, d\phi' \tag{20}$$

where x' denotes x as a function of ϕ'. The order of the two processes summation and integration cannot immediately be transposed, because the series would not be convergent. However, equation (20) may be regarded as the limit of the expression which is obtained by introducing r^n into the general term of the series, where r is a quantity less than 1 and in the limit is equal to 1. Thus, summing the series

$$\sum_{n=1}^{\infty} r^n \sin n\phi \cos n\phi'$$

leads to

$$x = \phi - \frac{\epsilon}{\pi} \int_0^{\pi} \left[\frac{r \sin(\phi' + \phi)}{1 - 2r \cos(\phi' + \phi) + r^2} - \frac{r \sin(\phi' - \phi)}{1 - 2r \cdot \cos(\phi' - \phi) + r^2}\right] \cos x' \, d\phi'$$

In the limit $r \rightarrow 1$,

$$x = \phi - \frac{\epsilon}{2\pi} \int_0^\pi \left[\cot \frac{1}{2}(\phi' + \phi) - \cot \frac{1}{2}(\phi' - \phi)\right] \cos x' \, d\phi'$$

and, after integrating by parts,

$$\phi - x = \frac{\epsilon}{\pi} \sin \phi \int_0^\pi \frac{\cos x' \, d\phi'}{\cos \phi' - \cos \phi} \tag{21}$$

This is a nonlinear integral equation closely related to the one derived by Theodorsen and Garrick as the basis of their arbitrary-airfoil theory. In fact, the procedure of solving it is precisely the one devised by Theodorsen and Garrick and described in detail in reference 1. Thus, a solution is sought, not by development in series according to some parameter, but by the following procedure of successive approximations:

Let ϕ_k be an approximation of the velocity potential found in any manner whatsoever. Inserting this approximation into the right-hand side of equation (21) results in an improved approximation ϕ_{k+1}. In the same manner, further approximations ϕ_{k+2}, ϕ_{k+3}, . . . are obtained until the iteration ceases, that is, until a pair of successive approximations is no longer distinguishable within the limits of the desired accuracy. Thus, equation (21) is written as follows:

$$\phi_{k+1} - x = \frac{\epsilon}{\pi} \sin \phi_k \int_0^\pi \frac{\cos x' \, d\phi_k'}{\cos \phi_k' - \cos \phi_k} \tag{22}$$

Following the idea of Stokes, ϕ_0 is taken equal to $x + A_1(\epsilon)\sin x$.

Then to the second order of approximation,

$$\phi_1 - x = \frac{\epsilon}{\pi} \sin \phi_0 \int_0^\pi \frac{\cos x' \, d\phi_0'}{\cos \phi_0' - \cos \phi_0} \tag{23}$$

Now, according to reference 9 (compare with eqs. (11a) and (12)), it follows that x as a function of ϕ_0 is given by

$$x = \phi_0 + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n J_n(nA_1)\sin n\phi_0 \qquad (A_1 \leqq 1)$$

and

$$\cos x = \frac{1}{2} A_1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[J_{n-1}(nA_1) - J_{n+1}(nA_1)\right]\cos n\phi_0$$

Hence, with the aid of the well-known definite integral

$$\int_0^{\pi} \frac{\cos n\phi' \; d\phi'}{\cos \phi' - \cos \phi} = \pi \frac{\sin n\phi}{\sin \phi}$$

equation (23) yields

$$\phi_1 - x = -\epsilon \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[J_{n-1}(nA_1) - J_{n+1}(nA_1)\right]\sin n\phi_0 \qquad (24)$$

or

$$\phi_1 = x - \epsilon \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[J_{n-1}(nA_1) - J_{n+1}(nA_1)\right]\sin n(x + A_1 \sin x)$$

From this point it is not feasible to continue this process in a completely analytical manner. For a given value of $A_1 \leqq 1$, however, it is a straightforward computation to tabulate corresponding values of x and ϕ_1 in the range 0 to π and, by means of a Fourier analysis, to express $\cos x$ as a cosine series in ϕ_1. (See eq. (25) for means of calculating the value of A_1 corresponding to a given value of ϵ.) Then, from equation (22), $\phi_2 - x$ is obtained as a sine series in ϕ_1. In constrast to the analytical development of the coefficients as power series in ϵ, this method for solving the integral equation (21) can be continued indefinitely without increase in the amount of computational labor for each step. It is clear that, in the limit $k \longrightarrow \infty$, this process yields equation (4a) but, in practice, it is terminated when the difference between two successive approximations is deemed negligible.

For values of $A_1(\epsilon)$ greater than unity, the approximation $\phi_0 = x + A_1(\epsilon)\sin x$ is not valid because x then becomes a multiple root of this equation in the range $0 \leqq x \leqq \pi$. The requirement is that ϕ be a continuously increasing function of x within this range at every stage of the iteration procedure. For $A_1 > 1$, a plausible beginning would be $\phi_0 = x + \sin x$; it may be presumed that this initial relationship between x and ϕ_0 leads to a convergent process up to a certain value of ϵ. The last valid relationship between x and ϕ would then again be taken as the beginning of the iteration procedure for a still higher range of values of ϵ. This process of continuation may presumably be carried on indefinitely.

Now, for some range of values $0 < A_1 < 1$ (but possibly for values of $\epsilon > 1$), equation (24) may be considered accurate enough to terminate the process of iteration. That is, approximately,

$$x = \phi + \epsilon \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[J_{n-1}(nA_1) - J_{n+1}(nA_1)\right]\sin n\phi$$

Comparison of this equation with equation (4a) shows that

$$A_n(\epsilon) = -\epsilon \frac{(-1)^n}{n}\left[J_{n-1}(nA_1) - J_{n+1}(nA_1)\right] \tag{25}$$

Note that for $n = 1$,

$$A_1(\epsilon) = \epsilon\left[J_0(A_1) - J_2(A_1)\right]$$

which provides a transcendental equation for the determination of A_1 for a given value of ϵ. With the use of the recurrence relation

$$J_{n-1}(nA_1) - J_{n+1}(nA_1) = \frac{2}{n}\frac{dJ_n(nA_1)}{dA_1}$$

it follows that

$$A_n(\epsilon) = -2\epsilon \frac{(-1)^n}{n^2} \frac{dJ_n(nA_1)}{dA_1}$$

Then

$$\sum_{n=1}^{\infty} A_{2n-1}(\epsilon) = 2\epsilon \frac{d}{dA_1} \sum_{n=1}^{\infty} \frac{J_{2n-1}\left[(2n - 1)A_1\right]}{(2n - 1)^2}$$

However,

$$\sum_{n=1}^{\infty} \frac{J_{2n-1}\left[(2n - 1)A_1\right]}{(2n - 1)^2} = \frac{1}{2} A_1$$

and hence (see eq. (8))

$$\sum_{n=1}^{\infty} A_{2n-1}(\epsilon) = \epsilon$$

Again (see eq. (9))

$$\frac{1}{2} A_0(\epsilon) = - \sum_{n=1}^{\infty} A_{2n}(\epsilon)$$

or

$$\frac{1}{2} A_0(\epsilon) = \frac{1}{2} \epsilon \frac{d}{dA_1} \sum_{n=1}^{\infty} \frac{J_{2n}(2nA_1)}{n^2}$$

But

$$\sum_{n=1}^{\infty} \frac{J_{2n}(2nA_1)}{n^2} = \frac{1}{2} A_1{}^2$$

and hence,

$$\frac{1}{2} A_0(\epsilon) = \frac{1}{2} \epsilon A_1$$

Then to the second order of approximation, equations (3) may be written as

$$\left.\begin{aligned} x &= \phi + \epsilon \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[J_{n-1}(nA_1) - J_{n+1}(nA_1)\right] e^{-n\psi}\sin n\phi \\ y &= \psi + \frac{1}{2} \epsilon A_1 - \epsilon \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[J_{n-1}(nA_1) - J_{n+1}(nA_1)\right] e^{-n\psi}\cos n\phi \quad (A_1 \leqq 1) \end{aligned}\right\} \tag{26}$$

It is of interest to examine the sum $S = \sum_{n=1}^{\infty} nA_n(\epsilon)\sin n\phi$ which appears in equation (14) for q^2. To the second order of approximation,

$$S = -\epsilon \sum_{n=1}^{\infty} (-1)^n \left[J_{n-1}(nA_1) - J_{n+1}(nA_1)\right] \sin n\phi \tag{27}$$

Now, by defining a new variable μ by the equation

$$\phi = \mu + A_1 \sin \mu$$

then

$$\mu = \phi + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n} J_n(nA_1)\sin n\phi$$

and

$$\cos \mu = \frac{1}{2} A_1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[J_{n-1}(nA_1) - J_{n+1}(nA_1) \right] \cos n\phi$$

Hence,

$$\sin \mu \frac{d\mu}{d\phi} = - \sum_{n=1}^{\infty} (-1)^n \left[J_{n-1}(nA_1) - J_{n+1}(nA_1) \right] \sin n\phi$$

or, since $\dfrac{d\mu}{d\phi} = \dfrac{1}{1 + A_1 \cos \mu}$,

$$S = \frac{\epsilon \sin \mu}{1 + A_1 \cos \mu} \qquad (A_1 \leqq 1;\ 0 \leqq \mu \leqq \pi)$$

The graph of S as a function of ϕ for the case $\epsilon = 0.70$ (and $A_1 = 0.606$ from eq. (25)) is shown as the solid curve in figure 2. The circles correspond to the evaluated series, equation (27), with $n = 1$ to 24. The values of A_n according to the second order of approximation are

$A_1 = 0.606$ $\quad A_2 = -0.164$ $\quad A_3 = 0.067$

$A_4 = -0.033$ $\quad A_5 = 0.017$ $\quad A_6 = -0.010$

$A_7 = 0.006$ $\quad \frac{1}{2} A_0 = 0.212$

These values agree remarkably well with those obtained by the method of Stokes. Note, however, that only one step was needed in the solution of integral equation (21) to achieve the accuracy of nine increasingly laborious steps according to powers of the first coefficient $A_1(\epsilon)$. The plus signs in figure 2 correspond to points computed with the Stokes values of A_n. The scattering of these points, particularly in the critical portion of the curve, is due to the slow convergence of the series for S with only the first few terms available and presages inaccurate determination of the derived quantities, fluid speed and pressure coefficient at the surface of the solid boundary. Here again the superiority of the integral-equation approach is demonstrated in that the second-order approximation provides an analytic form for the general coeffficient $A_n(\epsilon)$ (see eq. (25)). Table II lists the results of the calculation of fluid speed and pressure coefficient at the surface of the wavy wall to the second order of approximation, and figure 3 shows the corresponding graphs for an interval of one wave length.

In conclusion, the analysis of the present paper indicates the general undesirability of determining the velocity potential or the stream function in the form of a series whose coefficients are analytically developed in a power series of some parameter. Rather, the flow problem should be set up in the form of an integral equation whose solution can be obtained by a method of successive approximations. If this course is not possible, the parameter for the analytical development of the coefficients should be chosen so as to insure rapid convergence of the series. As the example of the present paper shows, this parameter may not be the most obvious one.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., November 25, 1953.

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TABLE I.- COEFFICIENTS a_{nm}

n	a_{nm}				
	m = 0	m = 1	m = 2	m = 3	m = 4
0		$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{11}{192}$	$-\frac{143}{4608}$
1	1	$-\frac{3}{8}$	$\frac{15}{64}$	$-\frac{1471}{9 \times 2^{10}}$	$\frac{82649}{45 \times 2^{14}}$
2	$-\frac{1}{2}$	$\frac{11}{24}$	$-\frac{53}{128}$	$\frac{16649}{45 \times 2^{10}}$	
3	$\frac{3}{8}$	$-\frac{215}{384}$	$\frac{683}{1024}$	$-\frac{262801}{45 \times 2^{13}}$	
4	$-\frac{1}{3}$	$\frac{1333}{15 \times 2^{7}}$	$-\frac{11873}{45 \times 2^{8}}$		
5	$\frac{125}{384}$	$-\frac{40187}{45 \times 2^{10}}$	$\frac{190849}{15 \times 2^{13}}$		
6	$-\frac{27}{80}$	$\frac{356983}{315 \times 2^{10}}$			
7	$\frac{7^{5}}{45 \times 2^{10}}$	$-\frac{4870981}{105 \times 2^{15}}$			
8	$-\frac{128}{315}$				
9	$\frac{9^{6}}{35 \times 2^{15}}$				

TABLE II.- CALCULATED VALUES OF FLUID SPEED AND PRESSURE COEFFICIENT

ϕ	μ	S	$\sum_{n=1}^{24} A_n \sin n\phi$	$\sum_{n=1}^{24} A_n \cos n\phi$	$\sum_{n=1}^{24} nA_n \sin n\phi$	$\sum_{n=1}^{24} nA_n \cos n\phi$	x	q	C_p
0	0	0	0	0.4879	0	0.401	0	1.670	-1.789
$\frac{\pi}{4}$	.496	.217	.3063	.4033	.218	.367	.479	1.494	-1.232
$\frac{\pi}{2}$	1.046	.465	.5523	.1385	.465	.238	1.019	1.120	-.254
$\frac{2\pi}{3}$	1.490	.665	.6300	-.1559	.665	.034	1.464	.852	.273
$\frac{3\pi}{4}$	1.760	.776	.6162	-.3445	.776	-.153	1.740	.720	.482
$\frac{5\pi}{6}$	2.093	.870	.5177	-.5610	.869	-.465	2.100	.587	.656
0.864π	2.222	.880	------	------	-----	------	-----	-----	------
$\frac{11\pi}{12}$	2.534	.795	.3513	-.7867	.795	-1.019	2.529	.461	.788
$\frac{47\pi}{48}$	2.977	.285	.1004	-.9026	.285	-1.501	2.976	.397	.842
π	π	0	0	-.9121	0	-1.550	π	.392	.846

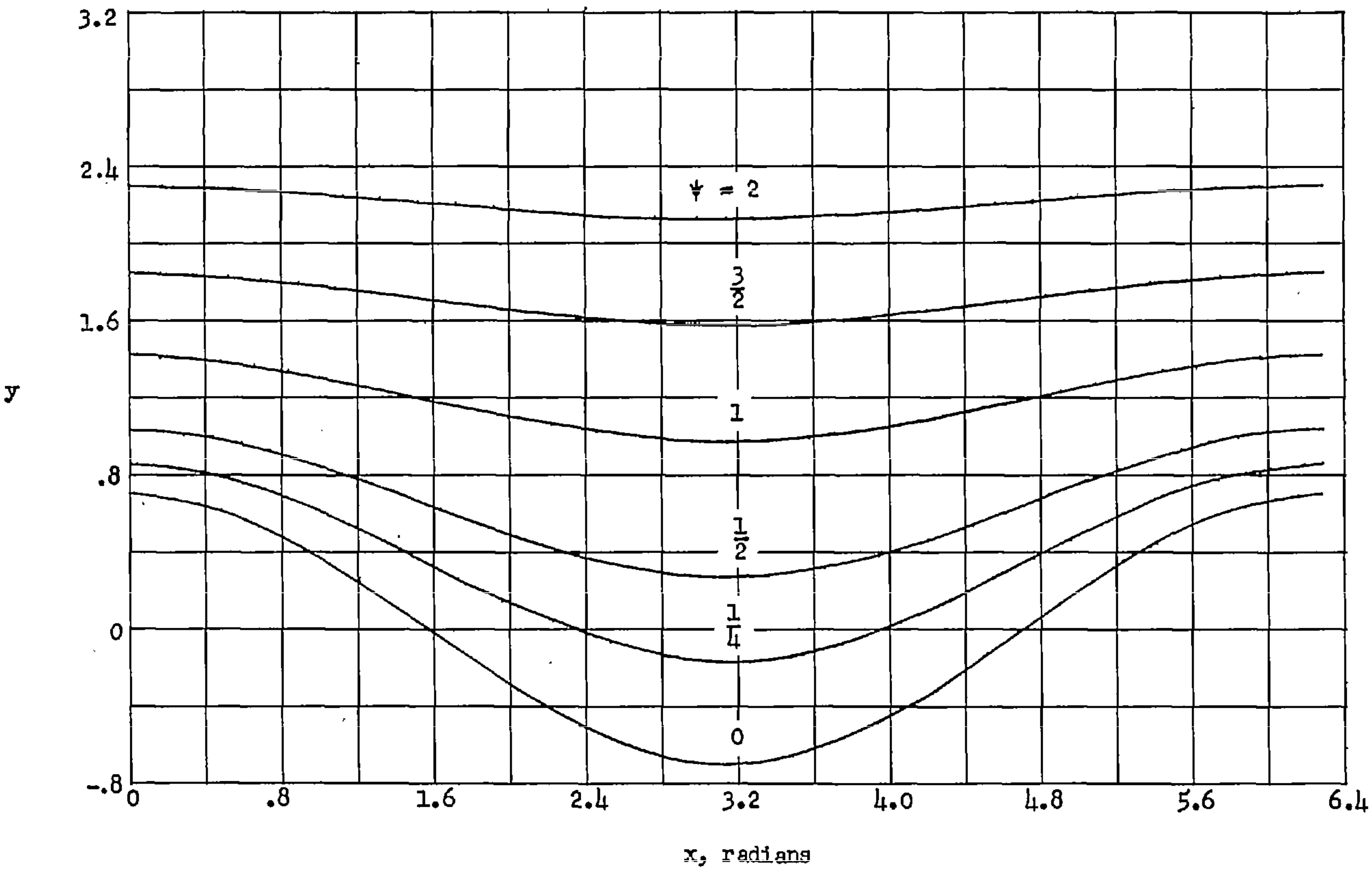

Figure 1.- Streamlines for $\epsilon = 0.70$.

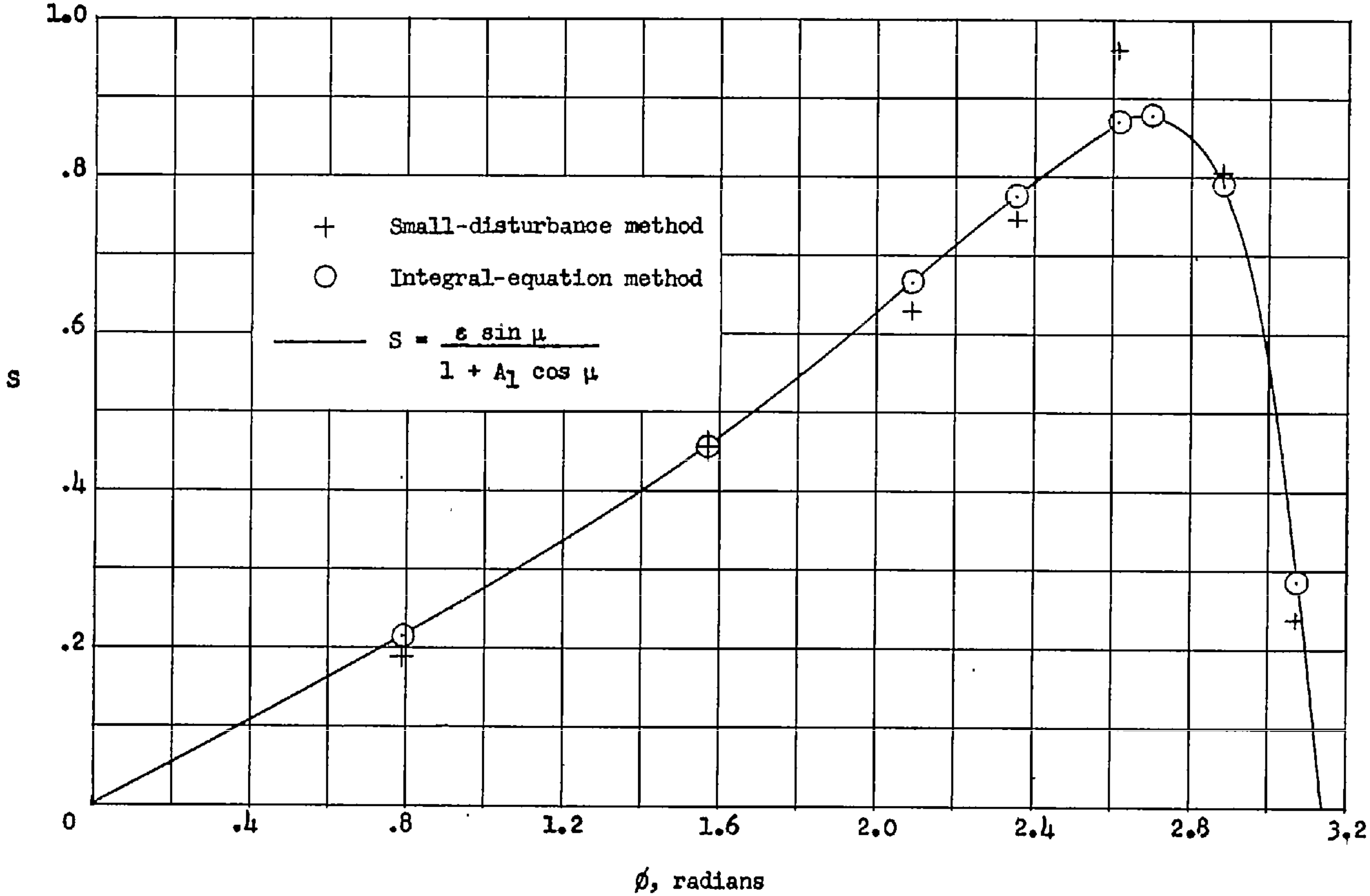

Figure 2.- Graph of $S = \dfrac{\epsilon \sin \mu}{1 + A_1 \cos \mu}$ as a function of ϕ.

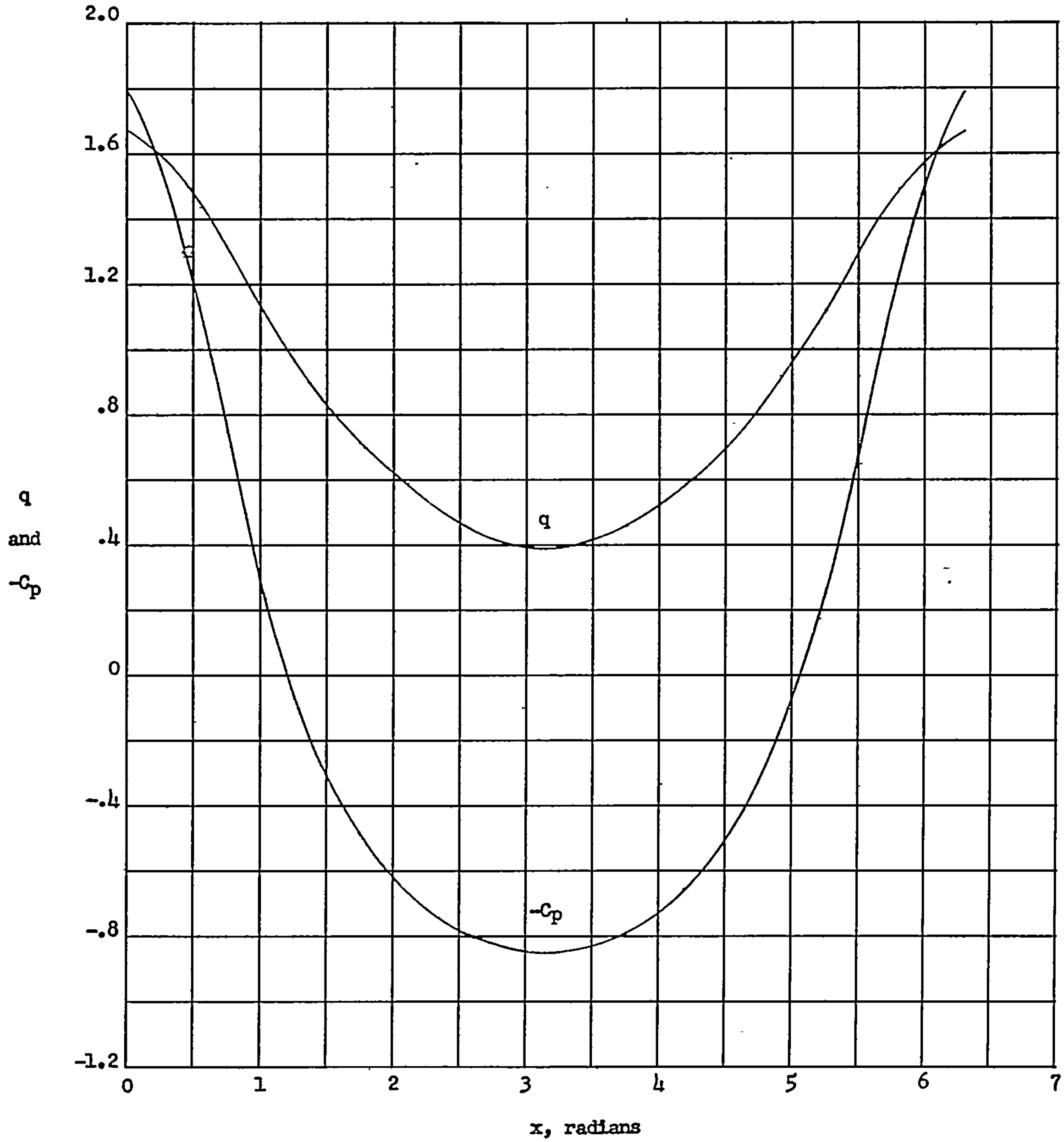

Figure 3.- Fluid speed and pressure coefficient at surface of wavy wall, $y = 0.70 \cos x$.

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